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NEW SUPER KdV SYSTEM WITH THE N=4 SCA AS THE HAMILTONIAN STRUCTURE

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Abstract

We present a new integrable extension of the $a = -2$, $N = 2$ SKdV hierarchy, with the "small" $N = 4$ superconformal algebra (SCA) as the second hamiltonian structure. As distinct from the previously known $N = 4$ supersymmetric KdV hierarchy associated with the same $N = 4$ SCA, the new system respects only $N = 2$ rigid supersymmetry. We give for it both matrix and scalar Lax formulations and consider its various integrable reductions which complete the list of known SKdV systems with the $N = 2$ SCA as the second hamiltonian structure. We construct a generalized Miura transformation which relates our system to the $\alpha = -2$, $N = 2$ super Boussinesq hierarchy and, respectively, the "small" $N = 4$ SCA to the $N = 2$ W_3 superalgebra.

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1 Introduction

In the recent years, supersymmetric extensions of integrable KP, KdV and NLS type hierarchies received much attention. This interest is motivated by both pure mathematical reasons and possible physical applications of these systems in non-perturbative $2D$ supergravity, matrix models, etc. One of the important motivations comes from the fact that these super-hierarchies are related, through their second hamiltonian structure, to superconformal algebras (both linear and W type ones). Thus they provide additional insights into the theory of W (super)algebras and conformal field theory. It is an urgent problem to fully classify all such systems and to reveal hidden relationships between them.

Up to now, most efforts were focused on studying $N=1$ and $N=2$ supersymmetric systems (see, e.g., [1- 15]). Recently, an example of hierarchy with higher supersymmetry was found, the $N=4$ SKdV hierarchy [16, 17]. It admits the "small" $N=4$ SCA (with $SU(2)$ affine subalgebra) as the hamiltonian structure. In refs. [18, 19] two different $N=2$ superfield scalar Lax formulations for this system were found.

In this letter we demonstrate the existence of one more integrable SKdV hierarchy with the same $N=4$ SCA as the second hamiltonian structure. Compared to the "genuine" $N=4$ SKdV, it possesses only $N=2$ rigid supersymmetry, and so does not possess a formulation in $N=4$ superspace. It should rather be viewed as an integrable extension of the $a=-2$, $N=2$ SKdV hierarchy by chiral and antichiral spin 1 $N=2$ superfields. One important difference with the $N=4$ SKdV system is that, like the $a=-2$, $N=2$ SKdV, it admits a matrix Lax formulation (on the superalgebra $sl(3|2)$), in parallel with the scalar one. Since $N=4$ supersymmetry is broken to $N=2$ in this "quasi" $N=4$ SKdV system, its reductions to the systems having different $N=2$ subalgebras of the $N=4$ SCA as the second hamiltonian structures yield non-equivalent hierarchies. In this way we find two new integrable systems, both having the $N=2$ SCA as the second hamiltonian structure. One of them possesses only rigid $N=1$ supersymmetry and no $U(1)$ symmetry. The second possesses no supersymmetry, but respects $U(1)$ symmetry. It is still different from the non-supersymmetric system of ref. [5]. Thus the "quasi" $N=4$ SKdV system allows us to enlarge, via its reductions, the list of known fermonic extensions of KdV associated with the $N=2$ SCA. An unexpected peculiarity of the system constructed is that it is related, through a generalized Miura transformation, to one of the three $N=2$ super Boussinesq hierarchies [10, 11]. This fact implies the existence of an intrinsic relationship between the "small" $N=4$ SCA and the nonlinear $N=2$ W_3 superalgebra.

2 Matrix and scalar Lax representations

We start by constructing the $N=2$ superfield matrix Lax operator which yields, through the appropriate Lax equation, the new SKdV system we intend to study in this paper. This will be done by applying the techniques of ref. [14] to the superalgebra $sl(3|2)$. We skip most details which will be given elsewhere. From now on, we deal with superfields on the $N=2$ superspace $X \equiv (x,\theta,\bar\theta)$ and use the following conventions

$$D=\frac{\partial}{\partial\theta}+\frac{1}{2}\bar\theta\frac{\partial}{\partial x},\quad \bar D=\frac{\partial}{\partial\bar\theta}+\frac{1}{2}\theta\frac{\partial}{\partial x},\quad \left\{D,\bar D\right\}=\frac{\partial}{\partial x},\quad D^2=\bar D^2=0\ . \tag{2.1}$$

In $N=2$ superspace, it is necessary to introduce two anticommuting Lax operators

$$\mathcal{L} = D + \Omega, \;\; \bar{\mathcal{L}} = \bar{D} + \bar{\Omega}, \tag{2.2}$$

where the connections Ω and $\bar{\Omega}$ take value in the loop algebra constructed from $sl(3|2)$. The loop parameter will be denoted by λ, and we choose the supertrace of a 5×5 matrix M to be

$$\text{str}M = M_{11} - M_{22} - M_{33} + M_{44} - M_{55}. \tag{2.3}$$

Then one can take:

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ DV & -V & \Phi_+ & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \;\; \bar{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -V & 0 & 0 & 1 & 0 \\ \Phi_- & 0 & 0 & 0 & 0 \\ -\bar{D}V & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

where V is an unconstrained bosonic $N=2$ superfield, Φ_+ and Φ_- are, respectively, chiral, $D\Phi_+ = 0$, and antichiral, $\bar{D}\Phi_- = 0$, bosonic superfields. These connections satisfy the zero curvature equations

$$D\Omega + \hat{\Omega}\Omega = 0, \;\; \bar{D}\bar{\Omega} + \hat{\bar{\Omega}}\bar{\Omega} = 0, \tag{2.5}$$

where $\hat{}$ denotes the automorphism of the superalgebra which reverses the sign of odd generators.

The Lax operators (2.2) may be submitted to the commuting evolution equations

$$\frac{\partial \mathcal{L}}{\partial t_k} = \mathcal{L}\mathcal{A}_k - \hat{\mathcal{A}}_k\mathcal{L}, \;\; \frac{\partial \bar{\mathcal{L}}}{\partial t_k} = \bar{\mathcal{L}}\mathcal{A}_k - \hat{\mathcal{A}}_k\bar{\mathcal{L}} \quad . \tag{2.6}$$

The general construction of the matrices $\mathcal{A}_k$ follows the same lines as in [14] and will not be given here. As an example which will be used in section 5, the explicit form of $\mathcal{A}_2$

$$\mathcal{A}_2 = - \begin{pmatrix} \lambda + \Phi_+\Phi_- & 0 & -\bar{D}\Phi_+ & 0 & 0 \\ -\Phi_+ D\Phi_- & \lambda + \Phi_+\Phi_- & \Phi_+' & 0 & 0 \\ D\Phi_-' - VD\Phi_- & -\Phi_-' & \Phi_+\Phi_- & -D\Phi_- & \Phi_- \\ \bar{D}\Phi_+ D\Phi_- & -\Phi_-\bar{D}\Phi_+ & -\bar{D}\Phi_+' - V\bar{D}\Phi_+ & \lambda + \Phi_+\Phi_- & 0 \\ 0 & 0 & \lambda\Phi_+ & 0 & \lambda \end{pmatrix}.$$

The equations (2.5), (2.6) are compatibility conditions for the linear problem

$$\mathcal{L}\Psi = 0, \;\; \bar{\mathcal{L}}\Psi = 0, \;\; \frac{\partial \Psi}{\partial t_k} + \mathcal{A}_k\Psi = 0, \tag{2.7}$$

where Ψ is a 5 component column vector. Through the usual elimination procedure, the first two of equations (2.7) lead to the following eigenvalue equation for Ψ_5

$$D\left(\partial_x + 2\,V - \Phi_+\partial^{-1}\Phi_-\right)\bar{D}\Psi_5 = \lambda\Psi_5. \tag{2.8}$$

Thus we arrive at the following $N=2$ scalar Lax operator and the Lax equation

$$L = D\left(\partial_x + 2\,V - \Phi_+\partial^{-1}\Phi_-\right)\bar{D}\,, \qquad \frac{\partial L}{\partial t_k} = -4\left[\, L^{\frac{k}{2}}_{\geq 0}, L \,\right]\,, \tag{2.9}$$

where the numerical coefficients and signs were chosen for further convenience.

The explicit form of the third and second flow equations will be given below. Here we only note that in the limit $\Phi_\pm = 0$ one recovers the $a = -2$, $N = 2$ SKdV hierarchy in the formulation which uses a chirality-preserving Lax operator. Such scalar Lax representations first appeared in ref. [9] and were studied in more detail in [18].

Thus the new $N = 2$ superhierarchy we have constructed is an integrable extension of the $a = -2$, $N = 2$ SKdV heirarchy by chiral and anti-chiral $N = 2$ superfields $\Phi_\pm$. In the next Section we will examine its hamiltonian structure.

Note that the even conserved charges are given by the following expression

$$H_k = \int \mu^{(2)} \ \mathrm{Res}\ L^{\frac{k}{2}} \tag{2.10}$$

where $\mu^{(2)} = dxd\theta d\bar{\theta}$ and the residue is defined as the coefficient before $D\bar{D}\partial^{-1}$.

3 Hamiltonian formulation

We could extract the second hamiltonian (or Poisson) structure associated with our system directly in the framework of the Lax representation, based on the formalism worked out in [18]. Here we do this in the hamiltonian framework.

It will be instructive to consider this system in parallel with the $N = 4$ SKdV system [16] which has the same $N = 2$ superfield content [17]. By introducing a parameter c (not to be confused with the central charge of the $N = 2$ and $N = 4$ SCA), we can uniformly write the third flow equations of both hierarchies as

$$\begin{aligned}
\frac{\partial V}{\partial t_3} &= -V''' + 3\ \left(\left[D,\bar{D}\right] V\ V\right)' + \frac{1}{2}(5-2c)\ \left(\left[D,\bar{D}\right] V^2\right)' \\
&\quad +2(c-3)\ \left(V^3\right)' + (c-1)\ \left(\Phi_-\Phi_+' - \Phi_+\Phi_-'\right)' \\
&\quad +6\left(V\Phi_+\Phi_-\right)' + \frac{1}{2}(c-4)\left(D\Phi_-\bar{D}\Phi_+\right)'\ , \qquad (3.1) \\
\frac{\partial \Phi_+}{\partial t_3} &= -c\ \Phi_+''' - 6\ D\left[\bar{D}\Phi_+ V' + \frac{1}{3}\ (c+2)\ \bar{D}\Phi_+' V\right. \\
&\quad \left. + \frac{4}{3}\ (4-c)\ V\bar{D}V\Phi_+ - \bar{D}\Phi_+\left(\Phi_+\Phi_- + \frac{1}{3}\ (c-7)\ V^2\right)\right]\ . \qquad (3.2)
\end{aligned}$$

(the equation for Φ_- can be restored through the discrete automorphism $\Phi_\pm \leftrightarrow \Phi_\mp, V \rightarrow -V, D \leftrightarrow \bar{D}$ which is a symmetry of both hierarchies). At $c = 4$ we get just the system constructed here; at $c = 1$ the manifestly $U(1)$ symmetric $a = 4, b = 0$ "gauge" of $N = 4$ SKdV in the $N = 2$ superfield form [17] is recovered. This notation emphasizes not only the similarity but also an essential difference between both systems: in the limit $\Phi_\pm = 0$, the second one goes into the $a = 4$, $N = 2$ SKdV system [1].

[1]Actually, the $a = -2$, $N = 2$ SKdV hierarchy can also be obtained as a consistent reduction of the $N = 4$ SKdV one, but with another choice of "gauge" with respect to the broken $SU(2)$ automorphism symmetry of $N = 4$ supersymmetry [17]. This property will be recovered in another context below.

Let us write the conserved dimension 3 hamiltonian for the system (3.1) - (3.2)

$$H_3^c = \int \mu^{(2)} \left\{ [D, \bar{D}]V\ V + \frac{2}{3}\,(c-3)V^3 + 2\ V\Phi_+\Phi_- + \frac{c}{2}\ \Phi_+'\Phi_- \right\} \tag{3.3}$$

(for our case we made use of the general formula (2.10), while in the $N = 4$ SKdV case it was given in ref. [17]). Now it is a matter of straightforward computation to show that the set (3.1) - (3.2) can be given the hamiltonian form

$$\frac{\partial V^A}{\partial t_3} = \{V^A, H_3^c\} = \mathcal{D}^{AB}\frac{\delta H_3^c}{\delta V^B}\ , \quad V^A \equiv (V, \Phi_-, \Phi_+)\ , \tag{3.4}$$

with the following Poisson brackets algebra

$$\left\{V^A(1), V^B(2)\right\} = \mathcal{D}^{AB}(1)\Delta^{(2)}(1-2)\ , \tag{3.5}$$

$$\begin{aligned}
&\mathcal{D}^{11} = \partial V - (\bar{D}V)\ D - (DV)\ \bar{D} - \frac{1}{2}\,[D, \bar{D}]\partial\ ,\\
&\mathcal{D}^{21} = \bar{D}\left(D\Phi_- + \Phi_- D\right)\ , \quad \mathcal{D}^{31} = D\left(\bar{D}\Phi_+ + \Phi_+\bar{D}\right)\ ,\\
&\mathcal{D}^{23} = 2\bar{D}\left(\partial - 2V\right)D\ , \quad \mathcal{D}^{22} = \mathcal{D}^{33} = 0\ .
\end{aligned} \tag{3.6}$$

Here $\Delta^{(2)}(1-2) = \delta(x_1 - x_2)(\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2)$. The remaining entries of the operator $\mathcal{D}^{AB}(1)$ can be easily deduced from those in (3.6).

The above Poisson superalgebra is the classical "small" $N = 4$ SCA (at some fixed value of the central charge) already used to construct the $N = 4$ SKdV hierarchy [16, 17]. Thus we have explicitly shown that the new hierarchy is also associated with this superalgebra as the second hamiltonian structure. The lack of rigid $N = 4$ supersymmetry in our $c = 4$ case and its presence in the $N = 4$ SKdV ($c = 1$) case can be readily established by considering the transformation properties of eqs. (3.1) - (3.2) and of the hamiltonian (3.3) under the hidden $N = 2$ supersymmetry [17]

$$\delta V = \frac{1}{2}\,\hat{\epsilon}\ D\Phi_- + \frac{1}{2}\,\hat{\bar{\epsilon}}\ \bar{D}\Phi_+\ , \quad \delta\Phi_- = -2\ \hat{\bar{\epsilon}}\bar{D}V\ , \quad \delta\Phi_+ = -2\ \hat{\epsilon}\ DV\ , \tag{3.7}$$

which extends the manifest $N = 2$ supersymmetry to $N = 4$. Under these variations the integrand in (3.3) is shifted by full spinor derivatives at $c = 1$. This is not the case for any other choice of c. The absence of $N = 4$ supersymmetry in the set (3.1) - (3.2) at $c = 4$ is obvious already from the fact that the linear terms in the r.h.s. of these equations appear with inequal coefficients. One more way to see the same property is to note that the generators of the hidden $N = 2$ supersymmetry

$$Q \sim \int \mu^{(2)}\bar{\theta}\Phi_-\ , \quad \bar{Q} \sim \int \mu^{(2)}\theta\Phi_+$$

are conserved at $c = 1$, but not at $c = 4$. To summarize, our system can be viewed as an integrable extension of the $a = -2$, $N = 2$ super KdV hierarchy, such that it preserves the $N = 2$ supersymmetry and $U(1)$ symmetry of the latter and possesses the "small" $N = 4$ SCA as the second hamiltonian structure.

One of the basic attributes of integrability is the existence of an infinite set of mutually commuting conserved charges. For the $N = 4$ SKdV this property has been first demonstrated

in [16], [17] by the explicit computation of the conserved bosonic charges up to the dimension 6 and proving that this system is bi-hamiltonian. Now, after constructing Lax operators for this system in refs. [19, 18], the existence of an infinite set of conserved quantities for it is a consequence of the appropriate Lax representation. The same is true of our "quasi" $N = 4$ super KdV system: the bosonic conserved quantities of any dimension can be computed according to the general formula (2.10) (or by a similar one in the matrix Lax formulation). To illustrate the general procedure, we quote the expressions for the conserved charges of dimension 2 and 4. For the purpose of comparing with the $N = 4$ SKdV case, we again write them in parallel for both systems

$$
\begin{aligned}
H_2^c &= \int \mu^{(2)} \left\{ \Phi_+\Phi_- + \frac{2}{3}(c-4)V^2 \right\} , \qquad (3.8) \\
H_4^c &= \int \mu^{(2)} \left\{ VD\Phi_-\bar{D}\Phi_+ + \frac{1}{4}\Phi_-^2\Phi_+^2 - \frac{1}{2}\Phi_-\Phi_+'' + \frac{2}{9}(4-c)\left[V^4 + \frac{1}{2}VV'' \right.\right. \\
&\quad \left.\left. - \frac{3}{2}V^2[D,\bar{D}]V - 3V^2\Phi_+\Phi_- + \frac{1}{2}\Phi_-\Phi_+'' \right] \right\} . \qquad (3.9)
\end{aligned}
$$

We see that, as opposed to the $N = 4$ SKdV case, the integrands in these charges at $c = 4$ are vanishing in the $N = 2$ SKdV limit $\Phi_\pm = 0$. Actually, this property extends to all the charges of even dimensions. It matches with the well-known fact that the $a = 4$, $N = 2$ SKdV possesses higher-order bosonic conserved quantities of all integer dimensions, while for the $a = -2$ system only odd dimension charges exist [5, 6].

Note that the $N = 4$ SKdV system, like the $a = 4$, $N = 2$ one, possesses a first hamiltonian structure which is local and linear, the charge $H_4^{c=1}$ being the relevant hamiltonian [16, 17]. It is obvious from the form of $H_4^{c=4}$ that no such first hamiltonian structure can be defined for the "quasi" $N = 4$ SKdV system, at least in the class of polynomial and local hamiltonians ($H_4^{c=4}$ contains no terms bilinear in V, so there is no way to reproduce the linear term in the r.h.s of eq. (3.1) at $c = 4$).

4 Integrable reductions

The interplay between the above $c = 1$ and $c = 4$ SKdV systems resembles the well-known relation between the $N = 1$ SKdV hierarchy [1, 3, 4] and the non-supersymmetric integrable fermionic extension of KdV constructed in [20, 3]. Despite the radically different symmetry properties, both these systems have $N = 1$ SCA as the second hamiltonian structure. They also admit a unifying parametrization by the parameter c, with $c = 1$ for the $N = 1$ supersymmetric system and $c = 4$ for the non-supersymmetric one. Analogously, the $N = 2$ SCA gives rise to two essentially different kinds of integrable extensions of KdV: three $N = 2$ supersymmetric ones and a non-supersymmetric one [5]. In our case we encounter a similar situation: two different integrable systems prove to be associated with the same $N = 4$ SCA as the second hamiltonian structure. One possesses full global $N = 4$ supersymmetry, while another has only $N = 2$ supersymmetry. We conjecture that this is a general phenomenon. Namely, for each of these superconformal algebras one can define a whole sequence of integrable fermionic extensions of KdV, ranging from the systems with the maximally possible global supersymmetry to the non-supersymmetric systems. If this conjecture is true, then, beginning from $N = 2$, the intermediate integrable systems should exist, in which the maximal supersymmetry is

broken only partially. The existence of the above “quasi” $N=4$ SKdV hierarchy with $N=2$ supersymmetry confirms this hypothesis. One may wonder why in the $N=2$ case only two kinds of integrable extensions of KdV are known. Now we wish to show that in fact two more integrable fermionic extensions of KdV equation with $N=2$ superconformal algebra as the second hamiltonian structure exist. Both are self-consistent reductions of our “quasi” $N=4$ super KdV system. The first of them possesses only $N=1$ supersymmetry and no internal $U(1)$ symmetry. The second yields a non-supersymmetric, though $U(1)$ symmetric, system which is still different from the extension constructed in ref. [5].

For our aim it will be convenient to deal with the equivalent $N=1$ superfield form of the “quasi” $N=4$ super KdV system. Passing to new θ's , $\theta^1=\frac{1}{2}(\theta+\bar{\theta})$, $\theta^2=\frac{1}{2}(\theta-\bar{\theta})$ (these are, respectively, real and imaginary with respect to the involution $\theta \leftrightarrow \bar{\theta}$), and redefining appropriately the spinor derivatives

$$D=\frac{1}{2}(D_1+D_2)\ ,\ \bar{D}=\frac{1}{2}(D_1-D_2),\ (D_1)^2=-(D_2)^2=\partial,\ \{D_1,D_2\}=0\ , \tag{4.1}$$

we can express the $N=2$ superfields $V(X), \Phi_\pm(X)$ through $N=1$ ones depending on (x,θ^1)

$$V=J_0+\theta^2\, G\ ,\ \Phi_\pm=J_\pm\mp\theta^2\, D_1J_\pm\ . \tag{4.2}$$

The system (3.1) - (3.2) at $c=4$ can be rewritten in $N=1$ superspace as follows (from here on, we omit the index 1 on $N=1$ spinor derivative and θ)

$$\begin{aligned}
\frac{\partial G}{\partial t_3} &= -G'''-3\,(GDG)'+6\left[G\left(J_+J_-+J_0^2\right)+J_0\,(J_+DJ_--J_-DJ_+)\right]' \\
&\quad +3\ \left(J_0'DJ_0-J_+DJ_-'-J_-DJ_+'+J_+'DJ_-+J_-'DJ_+\right)'\ , && (4.3)\\
\frac{\partial J_0}{\partial t_3} &= -J_0'''+2\left(J_0^3\right)'+3\left(J_-J_+'-J_+J_-'-G\ DJ_0\right)'+6\,(J_0J_+J_-)'\ , && (4.4)\\
\frac{\partial J_+}{\partial t_3} &= -4\ J_+'''-12\ J_0J_+''-6\ J_0'J_+'-6\ J_0^2J_+'+6\ DJ_+'DJ_0-3\ DJ_0'DJ_+ \\
&\quad +6\left(J_+J_-J_+'+J_+DJ_-DJ_+-J_0DJ_0DJ_+\right)+6\left(DJ_+'+J_0DJ_+\right)G \\
&\quad -3G'\ DJ_+ && (4.5)
\end{aligned}$$

(the equation for J_- can be obtained from that for J_+ through the appropriate involution). The $N=1$ superspace form of the Lax representation is as follows

$$L=\partial^2+J_0\partial+DJ_0D+GD-J_+\partial^{-1}DJ_-D\ ,\qquad \frac{\partial L}{\partial t_k}=-4\left[\ L^{\frac{k}{2}}_{>0},L\ \right]\ . \tag{4.6}$$

The hamiltonian (3.3) in terms of $N=1$ superfields reads

$$\begin{aligned}
H_3^c &= \frac{1}{2}\int dxd\theta\left\{GDG-DJ_0J_0'-\frac{c}{2}\left[(J_+'DJ_-+J_-'DJ_+\right]\right.\\
&\quad \left.+\,2G\left[(3-c)\ (J_0)^2-J_+J_-\right]+2J_0\left[J_-DJ_+-J_+DJ_-\right]\right\}\ . && (4.7)
\end{aligned}$$

The standard reduction $\Phi_\pm=0$ amounts to putting

$$J_\pm=0 \tag{4.8}$$

in (4.7). This yields the $a = 4$ and $a = -2$, $N = 2$ super KdV hierarchies for $c = 1$ and $c = 4$, respectively, with the same second hamiltonian structure $N = 2$ SCA generated by the $N = 1$ superfields G and J_0. However, one may embed the $N = 2$ SCA into the $N = 4$ SCA in different ways, and perform the reduction of our system so as to finally have another $N = 2$ subalgebra of $N = 4$ SCA as the hamiltonian structure. For the $c = 1$ case all such reductions yield $N = 2$ supersymmetric systems because the initial system is $N = 4$ supersymmetric. On the other hand, for the $c = 4$ case this is no longer true because the $c = 4$ system possesses only $N = 2$ rigid supersymmetry. It is just the one with respect to which G and J_0 form an irreducible multiplet and which is manifest in the formulation exposed in the previous sections. The $c = 4$ system is not invariant with respect to any other $N = 2$ subsymmetry of the whole rigid $N = 4$ supersymmetry formed by the manifest $N = 2$ supersymmetry transformatons and those given by (3.7).

If we wish to preserve the $N = 1$ superfield structure and hence $N = 1$ supersymmetry in the process of reduction, then, beside the $N = 2$ supersymmetry which is manifest in the previous $N = 2$ superfield formulation, only two other appropriate $N = 2$ supersymmetry subalgebras exist. They are formed by the explicit $N = 1$ supersymmetry transformations combined with the "real" or "imaginary" parts of the hidden $N = 2$ supersymmetry transformations (3.7). These parts correspond to singling out the following combinations of the parameters $\hat{\epsilon}$, $\bar{\hat{\epsilon}}$

$$\hat{\epsilon}_1 = \frac{1}{2}(\hat{\epsilon} + \bar{\hat{\epsilon}}) \ , \ \ \hat{\epsilon}_2 = \frac{1}{2}(\hat{\epsilon} - \bar{\hat{\epsilon}}) \ .$$

With respect to these two different $N = 2$ supersymmetries the $N = 1$ superfields

$$G, \ J_0, \ J_- \equiv J_1 + J_2, \ J_+ \equiv J_1 - J_2 \ ,$$

fall into the following two sets of $N = 2$ supermultiplets

$$(1). \ \ (G, \ J_1) \ , \ \ (J_0, \ J_2) \ ; \qquad (2). \ \ (G, \ J_2) \ , \ \ (J_0, \ J_1) \ . \tag{4.9}$$

It is easy to check that each of these pairs is indeed closed under the appropriate $N = 2$ supersymmetry, e.g.,

$$\delta_{\hat{\epsilon}_2} G = \hat{\epsilon}_2 J_1' \ , \ \delta_{\hat{\epsilon}_2} J_1 = -\hat{\epsilon}_2 G \ , \ \delta_{\hat{\epsilon}_2} J_0 = \hat{\epsilon}_2 D J_2 \ , \ \delta_{\hat{\epsilon}_2} J_2 = \hat{\epsilon}_2 D J_0 \ . \tag{4.10}$$

Let us now perform the reduction

$$J_0 = J_2 = 0 \ , \tag{4.11}$$

which brings (4.7) into

$$H^c_{red} = \int dx d\theta \left[\ GDG - c \ DJ_1 J_1' - 2 \ G(J_1)^2 \ \right] \ . \tag{4.12}$$

Note that, like the reduction (4.8), this reduction is self-consistent in the sense that both left- and right-hand sides of the equations for J_0 and J_2 disappear in this limit (for any flow with odd scaling dimension). Therefore, the resulting systems inherit the integrability properties of the initial systems, in particular, the presence of infinite sets of conserved charges. It can be easily checked that the Poisson brackets of the remaining superfields G, J_1 form the $N = 2$ SCA which is isomorphic to but different from the $N = 2$ SCA generated by G and J_0.

Looking at (4.12) we observe that at $c = 1$ this is none other than the hamiltonian of the $a = -2$, $N = 2$ SKdV hierarchy written in terms of $N = 1$ superfields. It is easy to check its invariance both under $N = 2$ supersymmetry (4.10) and $U(1)$ transformations. The latter are realized on G and J_1 as

$$\delta J_1 = \alpha\theta^1 G \ , \ \delta G = \alpha \frac{\partial}{\partial\theta^1} J_1 \ , \tag{4.13}$$

α being a constant parameter. The fact that the reductions (4.8) and (4.11) of the same $a = 4, b = 0$ $N = 4$ super KdV equation yield the $a = 4$ and $a = -2$, $N = 2$ SKdV equations demonstrates, in a slightly different fashion than in [17], that both these inequivalent $N = 2$ SKdV hierarchies are contained as particular solutions in the single $N = 4$ SKdV one. The $N = 2$ structures G, J_0 and G, J_1 and, respectively, the reductions (4.8) and (4.11) are related to each other by some global $SU(2)$ transformation. The fact that they give rise to different $N = 2$ KdV systems comes from the non-invariance of the $N = 4$ SKdV hamiltonian with respect to the full set of such $SU(2)$ rotations [16], [17].

At $c = 4$ the hamiltonian (4.7) respects neither $N = 2$ supersymmetry nor $U(1)$ invariance (4.13)

$$H^{c=4}_{red} = \int dx d\theta \left[\, GDG - 4 \; DJ_1 J_1' - 2 \; G (J_1)^2 \, \right] \ . \tag{4.14}$$

Its only residual symmetry is $N = 1$ supersymmetry. Thus we have derived a new integrable extension of KdV equation which still has $N = 2$ SCA as the second hamiltonian structure, enjoys $N = 1$ supersymmetry but possesses no $U(1)$ invariance. It is instructive to present the relevant equations which follow from (4.5) upon imposing (4.11)

$$\begin{aligned} \frac{\partial G}{\partial t_3} &= -G''' - 3 \; (DGG)' + 6 \; \left(GJ_1^2 + J_1' DJ_1 - J_1 DJ_1' \right)' & (4.15) \\ \frac{\partial J_1}{\partial t_3} &= -4 \; J_1''' + 2 \left(J_1^3 \right)' + 6 \; DJ_1' G + 3 DJ_1 G' \ . & (4.16) \end{aligned}$$

The appropriate $N = 1$ superfield Lax operator and Lax equations can be obtained by substituting (4.11) into (4.6) and restricting k in (4.6) to odd integers, $k = 2n + 1$.

The absence of the second supersymmetry stems from the fact that its generator $\int dx d\theta J_1$ is not conserved. An interesting property of this reduction is that the even dimension conserved charges H_2 and H_4 (3.8) and (3.9) disappear, like in the $c = 1$ case. This property extends to the whole sequence of the even dimension charges and is related to the invariance of our "quasi" $N = 4$ SKdV hierarchy under the discrete automorphism

$$J_0 \to -J_0, \; J_+ \to J_-, \; J_- \to J_+, \; G \to G \ . \tag{4.17}$$

It preserves the odd-dimension conserved charges but reverses the sign of the even-dimension ones. This property and the fact that the reduction (4.11) is a fixed point of this discrete symmetry, explain the vanishing of the even-dimension charges in the case at hand (like in the case of $a = -2$, $N = 2$ SKdV).

We can choose the $N = 2$ SCA in the underlying $N = 4$ SCA so that the reduction associated with this choice yields a system having *no supersymmetry at all.* Let us go to the component fields

$$G = \xi_1 + \theta^1 u \ , \ J_0 = j_0 + \theta^1 \xi_2 \ , \ J_1 = j_1 + \theta^1 \xi_3 \ , \ J_2 = j_2 + \theta^1 \xi_4 \ , \tag{4.18}$$

where all fields are functions of t and x. We wish to examine the multiplet structure of these fields with respect to the $N = 2$ supersymmetry (3.7) which is not a symmetry of the hamiltonian (4.7) at $c = 4$ (though it is at $c = 1$). With respect to the transformations with parameters $\hat{\epsilon_2}$ and $\hat{\epsilon_1}$ these fields are split, respectively, into the following irreducible $N = 1$ multiplets

$$N = 1\ (\hat{\epsilon_2}): \quad (\xi_3, u)\ ,\ \ (j_0, \xi_4)\ ,\ \ (j_1, \xi_1)\ ,\ \ (j_2, \xi_2) \tag{4.19}$$
$$N = 1\ (\hat{\epsilon_1}): \quad (\xi_4, u)\ ,\ \ (j_0, \xi_3)\ ,\ \ (j_2, \xi_1)\ ,\ \ (j_1, \xi_2)\ . \tag{4.20}$$

The first two and last two pairs in each sequence form $N = 2$ multiplets. One can check that the $N = 2$ multiplet

$$(j_0, \xi_3, \xi_4, u) \tag{4.21}$$

generates an $N = 2$ SCA. Then one can enforce the reduction which yields an extension of KdV system with this particular $N = 2$ SCA as the second hamiltonian structure

$$j_1 = j_2 = \xi_1 = \xi_2 = 0\ . \tag{4.22}$$

It is a simple exercise to be convinced that this reduction is also consistent: evolution equations for the fields in (4.22) are identically satisfied when we impose (4.22).

When $c = 1$, due to $N = 4$ supersymmetry of the hamiltonian, one ends up again with an $N = 2$ supersymmetric integrable system, namely, the $a = -2$, $N = 2$ SKdV. A radically different situation comes out when $c = 4$. It is easy to see that in this case (4.22) explicitly breaks the whole supersymmetry of hamiltonian (4.7), since these constraints are covariant under $\hat{\epsilon}_{1,2}$ supersymmetries only, which are not respected by this hamiltonian. At the same time, they are covariant under the $U(1)$ symmetry which mixes j_1 with j_2 and ξ_1 with ξ_2. As a result, the reduced system should also be $U(1)$ covariant. The reduced hamiltonian is as follows

$$H^{c=4}_{red'} = \frac{1}{2}\int dx \left\{u^2 + 4\,(\xi_3 - \xi_4)\,(\xi_3 + \xi_4)' - j_0' j_0' - 2u(j_0)^2 + 8 j_0 \xi_3 \xi_4\right\}. \tag{4.23}$$

Its $U(1)$ (or $SO(1,1)$, depending on which reality properties are ascribed to the fields) symmetry realized by proper rescalings of the fermionic fields is manifest.

Thus we have got one more new integrable model with $N = 2$ SCA as the second hamiltonian structure algebra. It possesses no supersymmetry but respects global $U(1)$ invariance. It differs from the bi-hamiltonian non-supersymmetric $U(1)$ invariant fermionic extension of KdV with the $N = 2$ SCA second hamiltonian structure found in [5]. The main difference between both systems is the equation for the $U(1)$ current. In the system of ref. [5] it satisfies the trivial equation $\frac{\partial j_0}{\partial t_3} = 0$, while in our case it satisfies the mKdV equation

$$\frac{\partial j_0}{\partial t_3} = -j_0''' + 2\,(j_0^3)'\ .$$

Thus the j_0 equation decouples in both systems, but in different ways. The change of variables [5] $q = u - (j_0)^2$, $\psi_\pm = \exp\{\pm\partial^{-1} j_0\}(\xi_3 \mp \xi_4)$ fully separates the j_0 and $q, \psi_\pm$ equations, the latter set proving to be the same as in ref. [5]. The analysis in [5] was essentially bound by requiring the existence of a bi-hamiltonian structure, while our system certainly possesses no local first hamiltonian structure. The $N = 1$ superfield Lax formulation (4.6) under the reduction (4.22) gives rise to the two independent component Lax operators

$$L^{(1)} = \partial^2 + 2 j_0 \partial\ , \qquad L^{(2)} = \partial^2 + q - \psi_+ \partial^{-1} \psi_-\ , \tag{4.24}$$

each producing its own hierarchy. After putting $j_0 = \xi_4 = 0$ this system, like the one constructed in [5], goes into the non-supersymmetric fermionic extension of KdV with the $N = 1$ SCA as the second hamiltonian structure [20, 3].

5 Relation to the $N = 2$ Boussinesq hierarchy

As the last topic, we quote a surprising relationship between our "quasi" $N = 4$ SKdV system and the $\alpha = -2$, $N = 2$ super Boussinesq hierarchy.

We will establish this relationship at the level of the second flows. For the "quasi" $N = 4$ SKdV the corresponding equations can be straightforwardly derived, e.g., through the Poisson structure (3.5), (3.6) with $H_2^{c=4}$ (3.8) as the hamiltonian

$$\frac{\partial V_A}{\partial t_2} = -\frac{1}{2}\{V_A, H_2\} \tag{5.1}$$

(the numerical coefficient was chosen for further convenience). Their explicit form is

$$\begin{aligned} \frac{\partial V}{\partial t_2} &= -(\Phi_+\Phi_-)' , \\ \frac{\partial \Phi_+}{\partial t_2} &= 2\left(\Phi_+'V + DV\bar{D}\Phi_+ + \frac{1}{2}\Phi_+''\right), \ \frac{\partial \Phi_-}{\partial t_2} = 2\left(\Phi_-'V + \bar{D}VD\Phi_- - \frac{1}{2}\Phi_-''\right). \end{aligned} \tag{5.2}$$

Now, let us assume that at least one of the superfields $\Phi_\pm$ is invertible (i.e. starts with a constant) and define the following Miura type transformations

$$\tilde{V}_1 = V + \frac{\Phi_+'}{\Phi_+} , \quad W_1 = \Phi_+\Phi_- - 2\left(\frac{\Phi_+'}{\Phi_+}V + DV\frac{\bar{D}\Phi_+}{\Phi_+} + \frac{1}{2}\frac{\Phi_+''}{\Phi_+}\right) \tag{5.3}$$

or

$$\tilde{V}_2 = V - \frac{\Phi_-'}{\Phi_-} , \quad W_2 = \Phi_+\Phi_- + 2\left(\frac{\Phi_-'}{\Phi_-}V + \bar{D}V\frac{D\Phi_-}{\Phi_-} - \frac{1}{2}\frac{\Phi_-''}{\Phi_-}\right) . \tag{5.4}$$

It is easy to show that the spin 1 superfield $\tilde{V}$ and the composite spin 2 W satisfy, as a consequence of eqs. (5.2), the following set of equations

$$\frac{\partial \tilde{V}}{\partial t_2} = -W' , \quad \frac{\partial W}{\partial t_2} = -[D,\bar{D}]W' + 2\left(\tilde{V}W' + D\tilde{V}\bar{D}W + \bar{D}\tilde{V}DW\right) . \tag{5.5}$$

This system is recognized as the second flow of the $\alpha = -2$ $N = 2$ Boussinesq hierarchy [10, 11] (in the classification of ref. [10]). The same relation can be established for any flow and for the relevant Lax operators. Instead of presenting it explicitly here (it is a particular case of a general relationship between different families of Lax operators in $N = 2$ superspace [18]), we will illustrate it on the examples of the conserved charges H_2, H_3 and H_4. Namely, the Miura transformations (5.3) or (5.4) map the expressions (3.8), (3.3) and (3.9) at $c = 4$ on the following ones (up to rescalings)

$$H_2^b = \int \mu^{(2)}W, \ H_3^b = \int \mu^{(2)}\left([D,\bar{D}]\tilde{V}\ \tilde{V} + \frac{2}{3}\tilde{V}^3 + 2\tilde{V}W\right), \ H_4^b = \int \mu^{(2)}W^2 . \tag{5.6}$$

These are just the conserved charges of the $\alpha = -2$, $N = 2$ Boussinesq hierarchy.

Thus we have explicitly constructed the generalized Miura map relating our "quasi" $N = 4$ SKdV hierarchy to one of $N = 2$ Boussinesq hierarchies. Such a map implies the existence of an intrinsic relationship between the second hamiltonian structures of both hierarchies: the "small" $N = 4$ SCA and the $N = 2$ W_3 algebra. Note that similar Miura type transformations were considered in refs. [12, 13, 15].

This relationship provides a link between the $\alpha = -2$, $N = 2$ Boussinesq and $a = -2$, $N = 2$ SKdV hierarchies. A consistent reduction of the "quasi" $N = 4$ SKdV hierarchy is

$$\Phi_- = 0, \;\; \Phi_+ = const \;, \quad \text{or} \quad \Phi_- = const, \;\; \Phi_+ = 0 \;. \tag{5.7}$$

All the even-dimension conserved charges vanish while the odd-dimension ones go into those of the $a = -2$, $N = 2$ SKdV hierarchy. The same conditions imply the vanishing of one of the composite spin 2 superfields $W_{1,2}$ introduced by eqs. (5.3), (5.4). So from the $N = 2$ Boussinesq viewpoint such a reduction amounts to putting equal to zero the superfield W. Thus $W = 0$ is a consistent reduction of $\alpha = -2$, $N = 2$ Boussinesq hierarchy and takes it into the $a = -2$, $N = 2$ SKdV one.

6 Conclusions

In this paper we have presented one more $N = 2$ SKdV hierarchy with the "small" $N = 4$ SCA as the second hamiltonian structure. We constructed for it both matrix and scalar Lax formulations. As compared to the "genuine" $N = 4$ SKdV hierarchy, the new one respects only $N = 2$ supersymmetry. Thanks to the partial breaking of $N = 4$ supersymmetry, there are possible non-trivial consistent reductions of this system which yield previously unknown SKdV hierarchies with lower supersymmetry. In this way we found two new fermionic extensions of the KdV hierarchy, both having the $N = 2$ SCA as the second hamiltonian structure. One of them possesses $N = 1$ supersymmetry, the second one is a new non-supersymmetric extension. The existence of such "horizontal" hierarchies of integrable SKdV systems having the same hamiltonian structure and ranging from maximally supersymmetric systems to the systems with completely broken supersymmetry seems to be a general phenomenon. If this conjecture is true, then there should exist even more SKdV hierarchies associated with $N = 4$ SCA, $N = 1$ supersymmetric and non-supersymmetric ones. It would be interesting to check this.

One more notable property of the system presented here is its intimate relationship with the $\alpha = -2$, $N = 2$ super Boussinesq one. We explicitly constructed a generalized Miura transformation mapping the first hierarchy on the second one. This map also relates the two relevant second hamiltonian structures, the "small" $N = 4$ SCA and the $N = 2$ W_3 superalgebra, thus revealing hidden links between these two superconformal algebras. One may think about possible implications of this remarkable relationship in $N = 2$ W_3 strings, say.

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